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# Deformations and extensions of modified $\lambda$-differential Lie-Yamaguti algebras*

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**Abstract:** The modified $\lambda$-differential Lie-Yamaguti algebras are considered, in which a modified $\lambda$-differential Lie-Yamaguti algebra consisting of a Lie-Yamaguti algebra and a modified $\lambda$-differential operator. First we introduce the representation of modified $\lambda$-differential Lie-Yamaguti algebras. Furthermore, we establish the cohomology of a modified $\lambda$-differential Lie-Yamaguti algebra with coefficients in a representation. Finally, we investigate the one-parameter formal deformations and Abelian extensions of modified $\lambda$-differential Lie-Yamaguti algebras using the second cohomology group.

**Key words:** Lie-Yamaguti algebra; modified $\lambda$-differential operator; representation and cohomology; one-parameter formal deformation; Abelian extension

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The concept of Lie-Yamaguti algebras (also called a general Lie triple system or a Lie triple algebra) was introduced by Kinyon et al. (2001) in the study of Courant algebroids, and can be constructed from Leibniz algebras, which originated from Nomizu's work on the invariant affine connections on homogeneous spaces in the 1950s (Nomizu, 1954). Its roots can be traced back to Yamaguti's study of the general Lie triple system (Yamaguti, 1958). Yamaguti established the representation and cohomology theory of Lie-Yamaguti algebras in Yamaguti(1967). Benito et al. (2005,2009) have studied Lie-Yamaguti algebras related to simple Lie algebras and irreducible Lie-Yamaguti algebras. The deformation and extension theory for Lie-Yamaguti algebras were investigated in Lin et al. (2015) and Zhang et al. (2015). Further research on Lie-Yamaguti algebras could be found in Guo(2023), Teng(2024), Guo et al. (2024) and references cited therein.

The notion of $\lambda$-differential algebras was first invented by Guo et al. (2008), which generalizes simultaneously the concept of the classical differential algebra and difference algebra. On the other hand, the term modified $r$-matrix stemmed from the concept of modified classical Yang-Baxter equation, which was introduced in the work of Semenov-Tyan-Shanskii(1983). Jiang et al. (2024) developed the deformation theory of modified $r$-matrices and cohomologies of related algebraic structures. Motivated by the modified $r$-matrices

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and $\lambda$-differential algebras, Peng and his collaborators introduced the concept of modified $\lambda$-differential Lie algebras (Peng et al. , 2022).

It is well known that Lie-Yamaguti algebras are a generalization of Lie algebras and Lie triple systems. It is very natural to investigate modified $\lambda$-differential Lie-Yamaguti algebras. This was our motivation for writing the present paper. Specifically, we define the representation and cohomology of the modified $\lambda$-differential Lie-Yamaguti algebras and apply them to the one-parameter formal deformation and Abelian extension of modified $\lambda$-differential Lie-Yamaguti algebras. All tensor products, vector spaces, and linear maps are over a field $\mathbb{K}$ of characteristic 0.

## 1 Representations of modified $\lambda$-differential Lie-Yamaguti algebras

In this section, we introduce the concept of modified $\lambda$-differential Lie-Yamaguti algebras. We explore the relationship between modified $\lambda$-differential operators and derivations, and provide some examples. Finally, we propose the representation of modified $\lambda$-differential Lie-Yamaguti algebras.

First we recall some basic definitions of Lie-Yamaguti algebras from Kinyon et al. (2001) and Yamaguti (1967).

**Definition 1**(Kinyon et al. , 2001) A Lie-Yamaguti algebra is a triple $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$, in which $\mathfrak{g}$ is a vector space together with a binary bracket $[\cdot,\cdot]$ and a ternary bracket $\{\cdot,\cdot,\cdot\}$ on $\mathfrak{g}$ satisfying

$$(\mathrm{LY1})\quad [x, y] = -[y, x],$$
$$(\mathrm{LY2})\quad \{x, y, z\} = -\{y, x, z\},$$
$$(\mathrm{LY3})\quad \circlearrowleft_{x,y,z}[[x, y], z] + \circlearrowleft_{x,y,z}\{x,y,z\} = 0,$$
$$(\mathrm{LY4})\quad \circlearrowleft_{x,y,z}\{[x, y], z, a\} = 0,$$
$$(\mathrm{LY5})\quad \{a, b, [x, y]\} = [\{a, b, x\}, y] + [x, \{a, b, y\}],$$
$$(\mathrm{LY6})\quad \{a, b, \{x,y, z\}\} = \{\{a, b, x\}, y, z\} + \{x, \{a, b, y\}, z\} + \{x, y, \{a, b, z\}\},$$

for any $x, y, z, a, b \in \mathfrak{g}$ and where $\circlearrowleft_{x,y,z}$ denotes the sum over cyclic permutation of $x, y, z$, that is $\circlearrowleft_{x,y,z}[[x, y], z] = [[x, y], z] + [[z, x], y] + [[y, z], x]$.

**Example 1** Let $(\mathfrak{g}, [\cdot,\cdot])$ be a Lie algebra. Define a ternary bracket $\{\cdot,\cdot,\cdot\}$ on $\mathfrak{g}$ by

$$\{x, y, z\} = [[x, y], z], \quad \forall x, y, z \in \mathfrak{g}. \tag{1}$$

Then $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$ is a Lie-Yamaguti algebra.

**Example 2** Let $(\mathfrak{g}, [\cdot,\cdot])$ be a Lie algebra with a reductive decomposition $\mathfrak{g} = \mathfrak{n}\oplus\mathfrak{m}$, i. e. $[\mathfrak{n},\mathfrak{n}]\subseteq \mathfrak{n}$ and $[\mathfrak{n},\mathfrak{m}]\subseteq \mathfrak{m}$. Define a bilinear bracket $[\cdot,\cdot]_{\mathfrak{m}}$ and a trilinear bracket $\{\cdot,\cdot,\cdot\}_{\mathfrak{m}}$ on $\mathfrak{m}$ by the projections of the Lie bracket:

$$[x, y]_{\mathfrak{m}} = \pi_{\mathfrak{m}}([x, y]), \qquad \{x, y, z\}_{\mathfrak{m}} = [\pi_{\mathfrak{n}}([x, y]), z], \quad \forall x, y, z \in \mathfrak{m},$$

where $\pi_{\mathfrak{n}}: \mathfrak{g} \to \mathfrak{n}$ and $\pi_{\mathfrak{m}}: \mathfrak{g} \to \mathfrak{m}$ are the projection maps. Then $(\mathfrak{m}, [\cdot,\cdot]_{\mathfrak{m}}, \{\cdot,\cdot,\cdot\}_{\mathfrak{m}})$ is a Lie-Yamaguti algebra.

**Example 3** Let $(\mathfrak{g}, \star)$ be a (left) Leibniz algebra. Define a binary and ternary bracket on $\mathfrak{g}$ by

$$[x, y] = x \star y - y \star x, \ \{x, y, z\} = -(x \star y) \star z, \quad \forall x, y, z \in \mathfrak{g}.$$

Then $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$ is a Lie-Yamaguti algebra.

**Example 4** Let $\mathfrak{g}$ be a 2-dimensional vector space with a basis $\varepsilon_1, \varepsilon_2$. If we define a binary non-zero bracket $[\cdot,\cdot]$ and a ternary non-zero bracket $\{\cdot,\cdot,\cdot\}$ on $\mathfrak{g}$ as follows:

$$[\varepsilon_1, \varepsilon_2] = -[\varepsilon_2, \varepsilon_1] = \varepsilon_1, \ \{\varepsilon_1, \varepsilon_2, \varepsilon_2\} = -\{\varepsilon_2, \varepsilon_1, \varepsilon_2\} = \varepsilon_1,$$

then $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$ is a Lie-Yamaguti algebra.

**Example 5** Let $\mathfrak{g}$ be a 3-dimensional vector space with a basis $\varepsilon_1, \varepsilon_2, \varepsilon_3$. If we define a binary non-zero bracket $[\cdot,\cdot]$ and a ternary non-zero bracket $\{\cdot,\cdot,\cdot\}$ on $\mathfrak{g}$ as follows:

$$[\varepsilon_1, \varepsilon_2] = -[\varepsilon_2, \varepsilon_1] = \varepsilon_3,\ \{\varepsilon_1, \varepsilon_2, \varepsilon_1\} = -\{\varepsilon_2, \varepsilon_1, \varepsilon_1\} = \varepsilon_3,$$

then $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$ is a Lie-Yamaguti algebra.

Next we introduce the concept of modified $\lambda$-differential Lie-Yamaguti algebras motivated by the modified $\lambda$-differential Lie algebras (Peng et al. , 2022).

**Definition 2** Let $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$ be a Lie-Yamaguti algebra, for $\lambda \in \mathbb{K}$, a linear map $\varphi: \mathfrak{g} \to \mathfrak{g}$ is said to a modified $\lambda$-differential operator if $\varphi$ satisfies

$$\varphi[x, y] = [\varphi(x), y] + [x, \varphi(y)] + \lambda[x, y], \tag{2}$$

$$\varphi\{x, y, z\} = \{\varphi(x), y, z\} + \{x, \varphi(y), z\} + \{x, y, \varphi(z)\} + 2\lambda\{x, y, z\}, \tag{3}$$

for any $x, y, z \in \mathfrak{g}$. Moreover, A modified $\lambda$-differential Lie-Yamaguti algebra is a quadruple $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \varphi)$ consisting of a Lie-Yamaguti algebra $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$ together with a modified $\lambda$-differential operator $\varphi$.

A homomorphism between two modified $\lambda$-differential Lie-Yamaguti algebras $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \varphi)$ and $(\mathfrak{g}', [\cdot,\cdot]', \{\cdot,\cdot,\cdot\}', \varphi')$ is a linear map $\eta: \mathfrak{g} \to \mathfrak{g}'$ satisfying

$$\eta([x, y]) = [\eta(x), \eta(y)]', \quad \eta(\{x, y, z\}) = \{\eta(x), \eta(y), \eta(z)\}' \text{ and } \varphi'(\eta(x)) = \eta(\varphi(x)),$$

for all $x, y, z \in \mathfrak{g}$. Furthermore, If $\eta$ is bijective, it is said that $\eta$ is an isomorphism from $\mathfrak{g}$ to $\mathfrak{g}'$.

**Remark 1** (i) Let $\varphi$ be a modified $\lambda$-differential operator on $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$. If $\lambda = 0$, then $\varphi$ is a derivation on $\mathfrak{g}$, and then $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \varphi)$ becomes a Lie-Yamaguti algebra with a derivation. See Guo(2023) for more details about Lie-Yamaguti algebras with a derivation.

(ii) When a Lie-Yamaguti algebra reduces to a Lie algebra, that is $\{\cdot,\cdot,\cdot\} = 0$, we get the notion of a modified $\lambda$-differential Lie algebra. See Peng et al. (2022) for more details about modified $\lambda$-differential Lie algebras.

(iii) If a Lie-Yamaguti algebra $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$ reduces to a Lie triple system $(\mathfrak{g}, \{\cdot,\cdot,\cdot\})$, we obtain the notion of a modified $\lambda$-differential operator on a Lie triple system $(\mathfrak{g}, \{\cdot,\cdot,\cdot\})$ (Long et al. , 2024).

**Example 6** Assume that $(\mathfrak{g}, [\cdot,\cdot], \varphi)$ is a modified $\lambda$-differential Lie algebra in Definition 2. 5 (Peng et al. , 2022), and from Example 1, then $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \varphi)$ is a modified $\lambda$-differential Lie-Yamaguti algebra, where $\{\cdot,\cdot,\cdot\}$ is given by Eq. (1).

**Example 7** If $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$ is a Lie-Yamaguti algebra, then $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \mathrm{Id}_{\mathfrak{g}})$ is a modified $(-1)$-differential Lie-Yamaguti algebra, where $\mathrm{Id}_{\mathfrak{g}}: \mathfrak{g} \to \mathfrak{g}$ is an identity map.

**Example 8** Let $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$ be the 2-dimensional Lie-Yamaguti algebra given in Example 4. Then the quadruple $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \varphi)$ is a 2-dimensional modified $\lambda$-differential Lie-Yamaguti algebra, where $\varphi = \begin{pmatrix} k & 0 \\ 0 & -\lambda \end{pmatrix}$, for $k \in \mathbb{K}$.

**Example 9** Let $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$ be the 3-dimensional Lie-Yamaguti algebra given in Example 5. Then the quadruple $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \varphi)$ is a 3-dimensional modified $\lambda$-differential Lie-Yamaguti algebra, where

$$\varphi = \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix},$$

for $k \in \mathbb{K}$.

**Proposition 1** Let $[\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}]$ be a Lie-Yamaguti algebra. A linear map $\varphi$ is a modified $\lambda$-differential operator if and only if the map $\varphi + \lambda \mathrm{Id}_{\mathfrak{g}}$ is a derivation on $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$.

**Proof** Equations (2) and (3) are equivalent to

$$(\varphi + \lambda \mathrm{Id}_{\mathfrak{g}})[x, y] = [(\varphi + \lambda \mathrm{Id}_{\mathfrak{g}})(x), y] + [x, (\varphi + \lambda \mathrm{Id}_{\mathfrak{g}})(y)],$$

$$(\varphi + \lambda \mathrm{Id}_{\mathfrak{g}})\{x, y, z\} = \{(\varphi + \lambda \mathrm{Id}_{\mathfrak{g}})(x), y, z\} + \{x, (\varphi + \lambda \mathrm{Id}_{\mathfrak{g}})(y), z\} + \{x, y, (\varphi + \lambda \mathrm{Id}_{\mathfrak{g}})(z)\}.$$

The proposition follows.

**Definition 3**(Yamaguti, 1967) Let $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$ be a Lie-Yamaguti algebra and $V$ be a vector space. A representation of $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$ on $V$ consists of a linear map $\rho: \mathfrak{g} \to \mathrm{End}(V)$ and two bilinear maps $D, \theta: \mathfrak{g} \times \mathfrak{g} \to \mathrm{End}(V)$ such that

(R1) $D(x, y) - \theta(y, x) + \theta(x, y) + \rho([x, y]) - \rho(x)\rho(y) + \rho(y)\rho(x) = 0,$

(R2) $D([x, y], z) + D([y, z], x) + D([z, x], y) = 0,$

(R3) $\theta([x, y], a) = \theta(x, a)\rho(y) - \theta(y, a)\rho(x),$

(R4) $D(a, b)\rho(x) = \rho(x)D(a, b) + \rho(\{a, b, x\}),$

(R5) $\theta(x, [a, b]) = \rho(a)\theta(x, b) - \rho(b)\theta(x, a),$

(R6) $D(a, b)\theta(x, y) = \theta(x, y)D(a, b) + \theta(\{a, b, x\}, y) + \theta(x, \{a, b, y\}),$

(R7) $\theta(a, \{x, y, z\}) = \theta(y, z)\theta(a, x) - \theta(x, z)\theta(a, y) + D(x, y)\theta(a, z),$

for all $x, y, z, a, b \in \mathfrak{g}$. In this case, we also call $V$ a $\mathfrak{g}$-module. We denote a representation by $(V; \rho, \theta, D)$.

It can be concluded from (R6) that

(R6)′ $D(a, b)D(x, y) = D(x, y)D(a, b) + D(\{a, b, x\}, y) + D(x, \{a, b, y\}).$

Let $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$ be a Lie-Yamaguti algebra. We define linear maps $\mathrm{ad}: \mathfrak{g} \to \mathrm{End}(\mathfrak{g}), \mathcal{L}, \mathcal{R}: \otimes^2 \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ by

$$\mathrm{ad}(x)(z) := [x, z], \qquad \mathcal{L}(x, y)(z) := \{x, y, z\}, \qquad \mathcal{R}(x, y)(z) := \{z, x, y\},$$

for all $x, y, z \in \mathfrak{g}$. Then $(\mathfrak{g}; \mathrm{ad}, \mathcal{L}, \mathcal{R})$ forms a representation of $\mathfrak{g}$ on itself, called the adjoint representation.

**Definition 4** A representation of the modified $\lambda$-differential Lie-Yamaguti algebra $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \varphi)$ is a quintuple $(V; \rho, \theta, D, \varphi_V)$ such that the following conditions are satisfied:

(i) $(V; \rho, \theta, D)$ is a representation of the Lie-Yamaguti algebra $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$;

(ii) $\varphi_V: V \to V$ is a linear map satisfying the following equations

$$\varphi_V(\rho(x)u) = \rho(\varphi(x))u + \rho(x)\varphi_V(u) + \lambda\rho(x)u, \tag{4}$$

$$\varphi_V(\theta(x, y)u) = \theta(\varphi(x), y)u + \theta(x, \varphi(y))u + \theta(x, y)\varphi_V(u) + 2\lambda\theta(x, y)u, \tag{5}$$

for any $x, y \in \mathfrak{g}$ and $u \in V$.

It can be concluded from Eq. (5) that

$$\varphi_V(D(x, y)u) = D(\varphi(x), y)u + D(x, \varphi(y))u + D(x, y)\varphi_V(u) + 2\lambda D(x, y)u. \tag{6}$$

**Example 10** $(\mathfrak{g}; \mathrm{ad}, \mathcal{L}, \mathcal{R}, \varphi)$ is an adjoint representation of the modified $\lambda$-differential Lie-Yamaguti algebra $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \varphi)$.

**Remark 2** Let $(V; \rho, \theta, D, \varphi_V)$ be a representation of the modified $\lambda$-differential Lie-Yamaguti algebra $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \varphi)$. If $\lambda = 0$, then $(V; \rho, \theta, D, \varphi_V)$ is a representation of a Lie-Yamaguti algebra with a derivation $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \varphi)$.

The following result exposes some connections between the representations of modified $\lambda$-differential Lie-Yamaguti algebras and Lie-Yamaguti algebras with derivations.

**Proposition 2** The quintuple $(V;\rho,\theta,D,\varphi_V)$ is a representation of the modified $\lambda$-differential Lie-Yamaguti algebra $(\mathfrak{g},[\cdot,\cdot],\{\cdot,\cdot,\cdot\},\varphi)$ if and only if $(V;\rho,\theta,D,\varphi_V+\lambda\mathrm{Id}_V)$ is a representation of a Lie-Yamaguti algebra with a derivation $(\mathfrak{g},[\cdot,\cdot],\{\cdot,\cdot,\cdot\},\varphi+\lambda\mathrm{Id}_{\mathfrak{g}})$.

**Proof** Equations (4)-(6) are equivalent to

$$(\varphi_V+\lambda\mathrm{Id}_V)(\rho(x)u)=\rho((\varphi+\lambda\mathrm{Id}_{\mathfrak{g}})(x))u+\rho(x)(\varphi_V+\lambda\mathrm{Id}_V)(u),$$
$$(\varphi_V+\lambda\mathrm{Id}_V)(\theta(x,y)u)=\theta((\varphi+\lambda\mathrm{Id}_{\mathfrak{g}})(x),y)u+\theta(x,(\varphi+\lambda\mathrm{Id}_{\mathfrak{g}})(y))u+\theta(x,y)(\varphi_V+\lambda\mathrm{Id}_V)(u),$$
$$(\varphi_V+\lambda\mathrm{Id}_V)(D(x,y)u)=D((\varphi+\lambda\mathrm{Id}_{\mathfrak{g}})(x),y)u+D(x,(\varphi+\lambda\mathrm{Id}_{\mathfrak{g}})(y))u+D(x,y)(\varphi_V+\lambda\mathrm{Id}_V)(u).$$

The proposition follows.

Next we construct the semidirect product of the modified $\lambda$-differential Lie-Yamaguti algebra.

**Proposition 3** If $(V;\rho,\theta,D,\varphi_V)$ is a representation of the modified $\lambda$-differential Lie-Yamaguti algebra $(\mathfrak{g},[\cdot,\cdot],\{\cdot,\cdot,\cdot\},\varphi)$, then $\mathfrak{g}\oplus V$ is a modified $\lambda$-differential Lie-Yamaguti algebra under the following maps:

$$[x+u,y+v]_{\ltimes}:=[x,y]+\rho(x)(v)-\rho(y)(u),$$
$$\{x+u,y+v,z+w\}_{\ltimes}:=\{x,y,z\}+D(x,y)(w)-\theta(x,z)(v)+\theta(y,z)(u),$$
$$\varphi\oplus\varphi_V(x+u):=\varphi(x)+\varphi_V(u),$$

for all $x,y,z\in\mathfrak{g}$ and $u,v,w\in V$. In the case, the modified $\lambda$-differential Lie-Yamaguti algebra $\mathfrak{g}\oplus V$ is called a semidirect product of $\mathfrak{g}$ and $V$, denoted by $\mathfrak{g}\ltimes V=(\mathfrak{g}\oplus V,[\cdot,\cdot]_{\ltimes},\{\cdot,\cdot,\cdot\}_{\ltimes},\varphi\oplus\varphi_V)$.

**Proof** In view of Yamaguti(1967), $(\mathfrak{g}\oplus V,[\cdot,\cdot]_{\ltimes},\{\cdot,\cdot,\cdot\}_{\ltimes})$ is a Lie-Yamaguti algebra. Next, for any $x,y,z\in\mathfrak{g}$ and $u,v,w\in V$, by eqs. (2)-(6), we have

$$\begin{aligned}\varphi\oplus\varphi_V[x+u,y+v]_{\ltimes}&=\varphi\oplus\varphi_V([x,y]+\rho(x)(v)-\rho(y)(u))=\varphi[x,y]+\varphi_V(\rho(x)(v)-\rho(y)(u))\\&=[\varphi(x),y]+[x,\varphi(y)]+\lambda[x,y]+\rho(\varphi(x))v+\rho(x)\varphi_V(v)+\lambda\rho(x)v\\&\quad-\rho(\varphi(y))u-\rho(y)\varphi_V(u)-\lambda\rho(y)u\\&=([\varphi(x),y]+\rho(\varphi(x))v-\rho(y)\varphi_V(u))+([x,\varphi(y)]+\rho(x)\varphi_V(v)-\rho(\varphi(y))u)\\&\quad+\lambda([x,y]+\rho(x)v-\rho(y)u)\\&=[\varphi\oplus\varphi_V(x+u),y+v]_{\ltimes}+[x+u,\varphi\oplus\varphi_V(y+v)]_{\ltimes}+\lambda[x+u,y+v]_{\ltimes},\end{aligned}$$

$$\begin{aligned}&\varphi\oplus\varphi_V\{x+u,y+v,z+w\}_{\ltimes}\\&=\varphi\{x,y,z\}+\varphi_V(D(x,y)(w)-\theta(x,z)(v)+\theta(y,z)(u))\\&=\{\varphi(x),y,z\}+\{x,\varphi(y),z\}+\{x,y,\varphi(z)\}+2\lambda\{x,y,z\}+D(\varphi(x),y)w+D(x,\varphi(y))w\\&\quad+D(x,y)\varphi_V(w)+2\lambda D(x,y)w-\theta(\varphi(x),z)v-\theta(x,\varphi(z))v-\theta(x,z)\varphi_V(v)-2\lambda\theta(x,z)v\\&\quad+\theta(\varphi(y),z)u+\theta(y,\varphi(z))u+\theta(y,z)\varphi_V(u)+2\lambda\theta(y,z)u\\&=(\{\varphi(x),y,z\}+D(\varphi(x),y)w-\theta(\varphi(x),z)v+\theta(y,z)\varphi_V(u))+(\{x,\varphi(y),z\}+D(x,\varphi(y))w\\&\quad-\theta(x,z)\varphi_V(v)+\theta(\varphi(y),z)u)+(\{x,y,\varphi(z)\}+D(x,y)\varphi_V(w)-\theta(x,\varphi(z))v+\theta(y,\varphi(z))u)\\&\quad+2\lambda(\{x,y,z\}+D(x,y)w-\theta(x,z)v+\theta(y,z)u)\\&=\{\varphi\oplus\varphi_V(x+u),y+v,z+w\}_{\ltimes}+\{x+u,\varphi\oplus\varphi_V(y+v),z+w\}_{\ltimes}\\&\quad+\{x+u,y+v,\varphi\oplus\varphi_V(z+w)\}_{\ltimes}+2\lambda\{x+u,y+v,z+w\}_{\ltimes}.\end{aligned}$$

So $(\mathfrak{g}\oplus V,[\cdot,\cdot]_{\ltimes},\{\cdot,\cdot,\cdot\}_{\ltimes},\varphi\oplus\varphi_V)$ is a modified $\lambda$-differential Lie-Yamaguti algebra.

## 2 Cohomology of modified $\lambda$-differential Lie-Yamaguti algebras

In this section, we will construct the cohomology of modified $\lambda$-differential Lie-Yamaguti algebras.

Firstly, let us recall the Yamaguti cohomology theory on Lie-Yamaguti algebras in Yamaguti(1967). Let $(V;\rho,\theta,D)$ be a representation of a Lie-Yamaguti algebra $(\mathfrak{g},[\cdot,\cdot],\{\cdot,\cdot,\cdot\})$, and we denote the set of $(n+1)$-cochains by $\mathcal{C}_{\mathrm{LY}}^{n+1}(\mathfrak{g},V)$, where

$$\mathcal{C}_{\mathrm{LY}}^{n+1}(\mathfrak{g},V)=\begin{cases}\mathrm{Hom}\big(\underbrace{\wedge^2\mathfrak{g}\otimes\cdots\otimes\wedge^2\mathfrak{g}}_{n},V\big)\times\mathrm{Hom}\big(\underbrace{\wedge^2\mathfrak{g}\otimes\cdots\otimes\wedge^2\mathfrak{g}}_{n}\otimes\mathfrak{g},V\big), & n\geqslant 1,\\ \mathrm{Hom}(\mathfrak{g},V), & n=0.\end{cases}$$

In the sequel, we recall the coboundary map of $(n+1)$-cochains on a Lie-Yamaguti algebra $\mathfrak{g}$ with the coefficients in the representation $(V;\rho,\theta,D)$.

If $n\geqslant 1$, for any $(f,g)\in\mathcal{C}_{\mathrm{LY}}^{n+1}(\mathfrak{g},V)$, $\mathfrak{K}_i=x_i\wedge y_i\in\wedge^2\mathfrak{g},(i=1,2,\cdots,n+1),z\in\mathfrak{g}$, the coboundary map $\delta^{n+1}=(\delta_{\mathrm{I}}^{n+1},\delta_{\mathrm{II}}^{n+1}):\mathcal{C}_{\mathrm{LY}}^{n+1}(\mathfrak{g},V)\to\mathcal{C}_{\mathrm{LY}}^{n+2}(\mathfrak{g},V),(f,g)\mapsto(\delta_{\mathrm{I}}^{n+1}(f,g),\delta_{\mathrm{II}}^{n+1}(f,g))$ is given as follows:

$$\begin{aligned}\delta_{\mathrm{I}}^{n+1}(f,g)(\mathfrak{K}_1,\cdots,\mathfrak{K}_{n+1})=&(-1)^n\big(\rho(x_{n+1})g(\mathfrak{K}_1,\cdots,\mathfrak{K}_n,y_{n+1})-\rho(y_{n+1})g(\mathfrak{K}_1,\cdots,\mathfrak{K}_n,x_{n+1})\\&-g(\mathfrak{K}_1,\cdots,\mathfrak{K}_n,[x_{n+1},y_{n+1}])\big)+\sum_{k=1}^{n}(-1)^{k+1}D(\mathfrak{K}_k)f(\mathfrak{K}_1,\cdots,\widehat{\mathfrak{K}_k},\cdots,\mathfrak{K}_{n+1})\\&+\sum_{1\leqslant k<l\leqslant n+1}(-1)^k f(\mathfrak{K}_1,\cdots,\widehat{\mathfrak{K}_k},\cdots,\{x_k,y_k,x_l\}\wedge y_l+x_l\wedge\{x_k,y_k,y_l\},\cdots,\mathfrak{K}_{n+1}),\end{aligned}$$

$$\begin{aligned}\delta_{\mathrm{II}}^{n+1}(f,g)(\mathfrak{K}_1,\cdots,\mathfrak{K}_{n+1},z)=&(-1)^n\big(\theta(y_{n+1},z)g(\mathfrak{K}_1,\cdots,\mathfrak{K}_n,x_{n+1})-\theta(x_{n+1},z)g(\mathfrak{K}_1,\cdots,\mathfrak{K}_n,y_{n+1})\big)\\&+\sum_{k=1}^{n+1}(-1)^{k+1}D(\mathfrak{K}_k)g(\mathfrak{K}_1,\cdots,\widehat{\mathfrak{K}_k},\cdots,\mathfrak{K}_{n+1},z)\\&+\sum_{1\leqslant k<l\leqslant n+1}(-1)^k g(\mathfrak{K}_1,\cdots,\widehat{\mathfrak{K}_k},\cdots,\{x_k,{}_k,x_l\}\wedge y_l+x_l\wedge\{x_k,y_k,y_l\},\cdots,\mathfrak{K}_{n+1},z)\\&+\sum_{k=1}^{n+1}(-1)^k g(\mathfrak{K}_1,\cdots,\widehat{\mathfrak{K}_k},\cdots,\mathfrak{K}_{n+1},\{x_k,y_k,z\}),\end{aligned}$$

where $\widehat{\ }$ denotes omission. For the case that $n=0$, for any $f\in\mathcal{C}_{\mathrm{LY}}^{1}(\mathfrak{g},V)$, the coboundary map $\delta^1=(\delta_{\mathrm{I}}^1,\delta_{\mathrm{II}}^1)$: $\mathcal{C}_{\mathrm{LY}}^{1}(\mathfrak{g},V)\to\mathcal{C}_{\mathrm{LY}}^{2}(\mathfrak{g},V),f\mapsto(\delta_{\mathrm{I}}^1(f),\delta_{\mathrm{II}}^1(f))$ is given by:

$$\delta_{\mathrm{I}}^1(f)(x,y)=\rho(x)f(y)-\rho(y)f(x)-f([x,y]),$$
$$\delta_{\mathrm{II}}^1(f)(x,y,z)=D(x,y)f(z)+\theta(y,z)f(x)-\theta(x,z)f(y)-f(\{x,y,z\}).$$

The corresponding cohomology groups are denoted by $\mathcal{H}_{\mathrm{LY}}^{\bullet}(\mathfrak{g},V)$.

Next we construct a cohomology theory for modified $\lambda$-differential Lie-Yamaguti algebras by using the Yamaguti cohomology.

Let $(\mathfrak{g},[\cdot,\cdot],\{\cdot,\cdot,\cdot\},\varphi)$ be a modified $\lambda$-differential Lie-Yamaguti algebra and $(V;\rho,\theta,D,\varphi_V)$ be a representation of it. For any $(f,g)\in\mathcal{C}_{\mathrm{LY}}^{n+1}(\mathfrak{g},V)$, $n\geqslant 1$, we define a linear map $\Phi^{n+1}=(\Phi_{\mathrm{I}}^{n+1},\Phi_{\mathrm{II}}^{n+1})$: $\mathcal{C}_{\mathrm{LY}}^{n+1}(\mathfrak{g},V)\to\mathcal{C}_{\mathrm{LY}}^{n+1}(\mathfrak{g},V)$, $(f,g)\mapsto(\Phi_{\mathrm{I}}^{n+1}(f),\Phi_{\mathrm{II}}^{n+1}(g))$ by:

$$\Phi_{\mathrm{I}}^{n+1}(f)=\sum_{i=1}^{2n}f\circ(\mathrm{Id}^{i-1},\varphi,\mathrm{Id}^{2n-i})\circ\big((\mathrm{Id}\wedge\mathrm{Id})^n\big)+(2n-1)\lambda f\circ\big((\mathrm{Id}\wedge\mathrm{Id})^n\big)-\varphi_V\circ f\circ\big((\mathrm{Id}\wedge\mathrm{Id})^n\big),$$

$$\Phi_{\mathrm{II}}^{n+1}(g)=\sum_{i=1}^{2n+1}g\circ(\mathrm{Id}^{i-1},\varphi,\mathrm{Id}^{2n+1-i})\circ\big((\mathrm{Id}\wedge\mathrm{Id})^n\wedge\mathrm{Id}\big)+2n\lambda g\circ\big((\mathrm{Id}\wedge\mathrm{Id})^n\wedge\mathrm{Id}\big)-\varphi_V\circ g\circ\big((\mathrm{Id}\wedge\mathrm{Id})^n\wedge\mathrm{Id}\big).$$

In particular, when $n=0$, define $\Phi^1:\mathcal{C}_{\mathrm{LY}}^{1}(\mathfrak{g},V)\to\mathcal{C}_{\mathrm{LY}}^{1}(\mathfrak{g},V)$ by $\Phi^1(f)=f\circ\varphi-\varphi_V\circ f$.

**Lemma 1** The map $\Phi^{n+1}\colon \mathcal{C}_{\mathrm{LY}}^{n+1}(\mathfrak{g}, V) \to \mathcal{C}_{\mathrm{LY}}^{n+1}(\mathfrak{g}, V)$ is a cochain map, that is, $\delta^{n} \circ \Phi^{n} = \Phi^{n+1} \circ \delta^{n}$.

**Proof** It follows by a straightforward and tedious calculations.

**Definition 5** Let $(V; \rho, \theta, D, \varphi_V)$ be a representation of the modified $\lambda$-differential Lie-Yamaguti algebra $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \varphi)$. We define the set of modified $\lambda$-differential Lie-Yamaguti algebra 1-cochains to be $\mathcal{C}_{\mathrm{MDLY}}^{1}(\mathfrak{g}, V) = \mathcal{C}_{\mathrm{LY}}^{1}(\mathfrak{g}, V)$. For $n \geqslant 1$, we define the set of modified $\lambda$-differential Lie-Yamaguti algebra $(n+1)$-cochains by

$$\mathcal{C}_{\mathrm{MDLY}}^{n+1}(\mathfrak{g}, V) := \mathcal{C}_{\mathrm{LY}}^{n+1}(\mathfrak{g}, V) \oplus \mathcal{C}_{\mathrm{LY}}^{n}(\mathfrak{g}, V).$$

Define a linear map $\partial^{1}\colon \mathcal{C}_{\mathrm{MDLY}}^{1}(\mathfrak{g}, V) \to \mathcal{C}_{\mathrm{MDLY}}^{2}(\mathfrak{g}, V)$ by

$$\partial^{1}(f_1) = \left(\delta^{1} f_1, -\Phi^{1}(f_1)\right), \ \forall f_1 \in \mathcal{C}_{\mathrm{MDLY}}^{1}(\mathfrak{g}, V).$$

For $n = 1$, we define the linear map $\partial^{2}\colon \mathcal{C}_{\mathrm{MDLY}}^{2}(\mathfrak{g}, V) \to \mathcal{C}_{\mathrm{MDLY}}^{3}(\mathfrak{g}, V)$ by

$$\partial^{2}\left((f_1, g_1), f_2\right) = \left(\delta^{2}(f_1, g_1), \delta^{1}(f_2) + \Phi^{2}(f_1, g_1)\right), \ \forall \left((f_1, g_1), f_2\right) \in \mathcal{C}_{\mathrm{MDLY}}^{2}(\mathfrak{g}, V).$$

Then, for $n \geqslant 2$, we define the linear map $\partial^{n+1}\colon \mathcal{C}_{\mathrm{MDLY}}^{n+1}(\mathfrak{g}, V) \to \mathcal{C}_{\mathrm{MDLY}}^{n+2}(\mathfrak{g}, V)$ by

$$\partial^{n+1}\left((f_1, g_1), (f_2, g_2)\right) = \left(\delta^{n+1}(f_1, g_1), \delta^{n}(f_2, g_2) + (-1)^{n+1}\Phi^{n+1}(f_1, g_1)\right),$$

for any $\left((f_1, g_1), (f_2, g_2)\right) \in \mathcal{C}_{\mathrm{MDLY}}^{n+1}(\mathfrak{g}, V)$.

In view of Lemma 1, we have the following theorem.

**Theorem 1** The map $\partial^{n+1}$ is a coboundary operator, i. e. , $\partial^{n+1} \circ \partial^{n} = 0$.

Therefore, from Theorem 1, we get a cochain complex $\left(\mathcal{C}_{\mathrm{MDLY}}^{\bullet}(\mathfrak{g}, V), \partial^{\bullet}\right)$ called the cochain complex of modified $\lambda$-differential Lie-Yamaguti algebra $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \varphi)$ with coefficients in $(V; \rho, \theta, D, \varphi_V)$. The cohomology of $\left(\mathcal{C}_{\mathrm{MDLY}}^{\bullet}(\mathfrak{g}, V), \partial^{\bullet}\right)$, denoted by $\mathcal{H}_{\mathrm{MDLY}}^{\bullet}(\mathfrak{g}, V)$, is called the cohomology of the modified $\lambda$-differential Lie-Yamaguti algebra $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \varphi)$ with coefficients in $(V; \rho, \theta, D, \varphi_V)$.

In particular, when $(V; \rho, \theta, D, \varphi_V) = (\mathfrak{g}; \mathrm{ad}, \mathcal{L}, \mathcal{R}, \varphi)$, we just denote $\left(\mathcal{C}_{\mathrm{MDLY}}^{*}(\mathfrak{g}, \mathfrak{g}), \partial^{\bullet}\right)$, $\mathcal{H}_{\mathrm{MDLY}}^{\bullet}(\mathfrak{g}, \mathfrak{g})$ by $\left(\mathcal{C}_{\mathrm{MDLY}}^{\bullet}(\mathfrak{g}), \partial^{\bullet}\right)$, $\mathcal{H}_{\mathrm{MDLY}}^{\bullet}(\mathfrak{g})$ respectively, and call them the cochain complex, the cohomology of modified $\lambda$-differential Lie-Yamaguti algebra $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \varphi)$ respectively.

It is obvious that there is a short exact sequence of cochain complexes:

$$0 \to \mathcal{C}_{\mathrm{LY}}^{\bullet-1}(\mathfrak{g}, V) \xrightarrow{in} \mathcal{C}_{\mathrm{MDLY}}^{\bullet}(\mathfrak{g}, V) \xrightarrow{pr} \mathcal{C}_{\mathrm{LY}}^{\bullet}(\mathfrak{g}, V) \to 0,$$

where $in$ and $pr$ are natural inclusion and projection respectively. It induces a long exact sequence of cohomology groups:

$$\cdots \to \mathcal{H}_{\mathrm{LY}}^{p-1}(\mathfrak{g}, V) \to \mathcal{H}_{\mathrm{MDLY}}^{p}(\mathfrak{g}, V) \to \mathcal{H}_{\mathrm{LY}}^{p}(\mathfrak{g}, V) \to \mathcal{H}_{\mathrm{MDLY}}^{p+1}(\mathfrak{g}, V) \to \mathcal{H}_{\mathrm{LY}}^{p+1}(\mathfrak{g}, V) \to \cdots.$$

## 3 One-parameter formal deformations of modified λ-differential Lie-Yamaguti algebras

In this section, we study one-parameter formal deformations of modified $\lambda$-differential Lie-Yamaguti algebras. Let $\mathbb{K}[[t]]$ be a ring of power series of one variable $t$, and let $\mathfrak{g}[[t]]$ be the set of formal power series over $\mathfrak{g}$. If $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$ is a Lie-Yamaguti algebra, then there is a Lie-Yamaguti algebra structure over the ring $\mathbb{K}[[t]]$ on $\mathfrak{g}[[t]]$ given by

$$\left[\sum_{i=0}^{\infty} x_i t^i, \sum_{j=0}^{\infty} y_j t^j\right] = \sum_{s=0}^{\infty} \sum_{i+j=s} [x_i, y_j] t^s, \ \left\{\sum_{i=0}^{\infty} x_i t^i, \sum_{j=0}^{\infty} y_j t^j, \sum_{k=0}^{\infty} z_k t^k\right\} = \sum_{s=0}^{\infty} \sum_{i+j+k=s} \{x_i, y_j, z_k\} t^s.$$

**Definition 6** A one-parameter formal deformation of the modified $\lambda$-differential Lie-Yamaguti algebra

$(\mathfrak{g},[\cdot,\cdot],\{\cdot,\cdot,\cdot\},\varphi)$ is a triple $(f_t,g_t,\varphi_t)$ of the forms

$$f_t=\sum_{i=0}^{\infty}f_it^i,\qquad g_t=\sum_{i=0}^{\infty}g_it^i,\qquad \varphi_t=\sum_{i=0}^{\infty}\varphi_it^i,$$

such that the following conditions are satisfied:

(i) $((f_i,\ g_i),\varphi_i)\in\mathcal{C}^2_{\mathrm{MDLY}}(\mathfrak{g}),f_0=[\cdot,\cdot],g_0=\{\cdot,\cdot,\cdot\}$ and $\varphi_0=\varphi$;

(ii) $(\mathfrak{g}[[t]],f_t,g_t,\varphi_t)$ is a modified $\lambda$-differential Lie-Yamaguti algebra over $\mathbb{K}[[t]]$.

Let $(f_t,g_t,\varphi_t)$ be a one-parameter formal deformation as above. Then, for any $x,y,z,a,b\in\mathfrak{g}$, the following equations must hold:

$$\begin{aligned}
&f_t(x,y)+f_t(y,x)=0,\quad g_t(x,y,z)+g_t(y,x,z)=0,\\
&\circlearrowleft_{x,y,z}f_t(f_t(x,y),z)+\circlearrowleft_{x,y,z}g_t(x,y,z)=0,\\
&\circlearrowleft_{x,y,z}g_t(f_t(x,y),z,a)=0,\\
&g_t(a,b,f_t(x,y))=f_t(g_t(a,b,x),y)+f_t(x,g_t(a,b,y)),\\
&g_t(a,b,g_t(x,y,z))=g_t(g_t(a,b,x),y,z)+g_t(x,g_t(a,b,y),z)+g_t(x,y,g_t(a,b,z)),\\
&\varphi_t(f_t(x,y))=f_t(\varphi_t(x),y)+f_t(x,\varphi_t(y))+\lambda f_t(x,y),\\
&\varphi_t(g_t(x,y,z))=g_t(\varphi_t(x),y,z)+g_t(x,\varphi_t(y),z)+g_t(x,y,\varphi_t(z))+2\lambda g_t(x,y,z).
\end{aligned}$$

Collecting the coefficients of $t^n$, we get that the above equations are equivalent to the following equations.

$$f_n(x,y)+f_n(y,x)=0,\ \ g_n(x,y,z)+g_n(y,x,z)=0,\tag{7}$$

$$\sum_{i=0}^{n}\circlearrowleft_{x,y,z}f_i(f_{n-i}(x,y),z)+\circlearrowleft_{x,y,z}g_n(x,y,z)=0,\tag{8}$$

$$\sum_{i=0}^{n}\circlearrowleft_{x,y,z}g_i(f_{n-i}(x,y),z,a)=0,\tag{9}$$

$$\sum_{i=0}^{n}g_i(a,b,f_{n-i}(x,y))=\sum_{i=0}^{n}f_i(g_{n-i}(a,b,x),y)+\sum_{i=0}^{n}f_i(x,g_{n-i}(a,b,y)),\tag{10}$$

$$\sum_{i=0}^{n}g_i(a,b,g_{n-i}(x,y,z))=\sum_{i=0}^{n}\Big(g_i(g_{n-i}(a,b,x),y,z)+g_i(x,g_{n-i}(a,b,y),z)+g_i(x,y,g_{n-i}(a,b,z))\Big),\tag{11}$$

$$\sum_{i=0}^{n}\varphi_i(f_{n-i}(x,y))=\sum_{i=0}^{n}\Big(f_i(\varphi_{n-i}(x),y)+f_i(x,\varphi_{n-i}(y))\Big)+\lambda f_n(x,y),\tag{12}$$

$$\sum_{i=0}^{n}\varphi_i(g_{n-i}(x,y,z))=\sum_{i=0}^{n}\Big(g_i(\varphi_{n-i}(x),y,z)+g_i(x,\varphi_{n-i}(y),z)+g_i(x,y,\varphi_{n-i}(z))\Big)+2\lambda g_n(x,y,z).\tag{13}$$

Note that for $n=0$, equations (7)-(13) are equivalent to $(\mathfrak{g},f_0,g_0,\varphi_0)$ is a modified $\lambda$-differential Lie-Yamaguti algebra.

**Proposition 4** Let $(f_t,g_t,\varphi_t)$ be a one-parameter formal deformation of a modified $\lambda$-differential Lie-Yamaguti algebra $(\mathfrak{g},[\cdot,\cdot],\{\cdot,\cdot,\cdot\},\varphi)$. Then $((f_1,g_1),\varphi_1)$ is a 2-cocycle in the cochain complex $(\mathcal{C}^{\bullet}_{\mathrm{MDLY}}(\mathfrak{g}),\partial^{\bullet})$.

**Proof** For $n=1$, equations (7)-(11) become

$$f_1(x,y)+f_1(y,x)=0,\ \ g_1(x,y,z)+g_1(y,x,z)=0,\tag{14}$$

$$\begin{aligned}&[f_1(x,y),z]+f_1([x,y],z)+[f_1(z,x),y]+f_1([z,x],y)+[f_1(y,z),x]+f_1([y,z],x)\\&\quad+g_1(x,y,z)+g_1(z,x,y)+g_1(y,z,x)=0,\end{aligned}\tag{15}$$

$$g_1([x,y],z,a)+\{f_1(x,y),z,a\}+g_1([z,x],y,a)+\{f_1(z,x),y,a\}+g_1([y,z],x,a)+\{f_1(y,z),x,a\}=0,\tag{16}$$

$$g_1(a,b,[x,y])+\{a,b,f_1(x,y)\}=f_1(\{a,b,x\},y)+[g_1(a,b,x),y]+f_1(x,\{a,b,y\})+[x,g_1(a,b,y)],\tag{17}$$

$$g_1(a,b,\{x,y,z\})+\{a,b,g_1(x,y,z)\}=g_1(\{a,b,x\},y,z)+\{g_1(a,b,x),y,z\}+g_1(x,\{a,b,y\},z)$$
$$+\{x,g_1(a,b,y),z\}+g_1(x,y,\{a,b,z\})+\{x,y,g_1(a,b,z)\}. \quad (18)$$

From eqs. (14)-(18), we get $(f_1,g_1)\in\mathcal{C}^2_{\mathrm{LY}}(\mathfrak{g})$ and $\delta^2(f_1,g_1)=0$. For $n=1$, equations (12) and (13) become

$$\varphi(f_1(x,y))+\varphi_1([x,y])=[\varphi_1(x),y]+f_1(\varphi(x),y)+[x,\varphi_1(y)]+f_1(x,\varphi(y))+\lambda f_1(x,y), \quad (19)$$

$$\varphi(g_1(x,y,z))+\varphi_1(\{x,y,z\})=\{\varphi_1(x),y,z\}+g_1(\varphi(x),y,z)+\{x,\varphi_1(y),z\}+g_1(x,\varphi(y),z)$$
$$+\{x,y,\varphi_1(z)\}+g_1(x,y,\varphi(z))+2\lambda g_1(x,y,z). \quad (20)$$

Further by eqs. (19) and (20), we have $\delta^1_{\mathrm{I}}(\varphi_1)+\Phi^2_{\mathrm{I}}(f_1)=0$ and $\delta^1_{\mathrm{II}}(\varphi_1)+\Phi^2_{\mathrm{II}}(g_1)=0$ respectively. Hence, $\partial^2((f_1,g_1),\varphi_1)=0$, that is, $((f_1,g_1),\varphi_1)$ is a 2-cocycle in $(\mathcal{C}^{\bullet}_{\mathrm{MDLY}}(\mathfrak{g}),\partial^{\bullet})$.

**Definition 7** The 2-cocycle $((f_1,g_1),\varphi_1)$ is called the infinitesimal of the one-parameter formal deformation $(f_t,g_t,\varphi_t)$ of a modified $\lambda$-differential Lie-Yamaguti algebra $(\mathfrak{g},[\cdot,\cdot],\{\cdot,\cdot,\cdot\},\varphi)$.

**Definition 8** Let $(f_t,g_t,\varphi_t)$ and $(f_t',g_t',\varphi_t')$ be two one-parameter formal deformations of a modified $\lambda$-differential Lie-Yamaguti algebra $(\mathfrak{g},[\cdot,\cdot],\{\cdot,\cdot,\cdot\},\varphi)$. A formal isomorphism from $(\mathfrak{g}[[t]],f_t,g_t,\varphi_t)$ to $(\mathfrak{g}[[t]],f_t',g_t',\varphi_t')$ is a power series $\Psi_t=\sum_{i=0}^{\infty}\Psi_i t^i:\mathfrak{g}[[t]]\to\mathfrak{g}[[t]]$, where $\Psi_i:\mathfrak{g}\to\mathfrak{g}$ are linear maps with $\Psi_0=\mathrm{Id}_{\mathfrak{g}}$, such that

$$\Psi_t\circ f_t=f_t'\circ(\Psi_t\otimes\Psi_t),\quad \Psi_t\circ g_t=g_t'\circ(\Psi_t\otimes\Psi_t\otimes\Psi_t),\quad \Psi_t\circ\varphi_t=\varphi_t'\circ\Psi_t. \quad (21)$$

In this case, we say that the two one-parameter formal deformations $(f_t,g_t,\varphi_t)$ and $(f_t',g_t',\varphi_t')$ are equivalent.

**Proposition 5** The infinitesimals of two equivalent one-parameter formal deformations of $(\mathfrak{g},[\cdot,\cdot],\{\cdot,\cdot,\cdot\},\varphi)$ are in the same cohomology class in $\mathcal{H}^2_{\mathrm{MDLY}}(\mathfrak{g})$.

**Proof** Let $\Psi_t:(\mathfrak{g}[[t]],f_t,g_t,\varphi_t)\to(\mathfrak{g}[[t]],f_t',g_t',\varphi_t')$ be a formal isomorphism. By expanding eqs. (21) and comparing the coefficients of $t$ on both sides, we have

$$f_1-f_1'=f_0\circ(\Psi_1\circ\mathrm{Id}_{\mathfrak{g}})+f_0\circ(\mathrm{Id}_{\mathfrak{g}}\circ\Psi_1)-\Psi_1\circ f_0,$$
$$g_1-g_1'=g_0\circ(\Psi_1\circ\mathrm{Id}_{\mathfrak{g}}\circ\mathrm{Id}_{\mathfrak{g}})+g_0\circ(\mathrm{Id}_{\mathfrak{g}}\circ\mathrm{Id}_{\mathfrak{g}}\circ\Psi_1)+g_0\circ(\mathrm{Id}_{\mathfrak{g}}\circ\Psi_1\circ\mathrm{Id}_{\mathfrak{g}})-\Psi_1\circ g_0,$$
$$\varphi_1-\varphi_1'=\varphi\circ\Psi_1-\Psi_1\circ\varphi.$$

That is, we can get

$$((f_1,g_1),\varphi_1)-((f_1',g_1'),\varphi_1')=(\delta^1(\Psi_1),-\Phi^1(\Psi_1))=\partial^1(\Psi_1)\in\mathcal{B}^2_{\mathrm{MDLY}}(\mathfrak{g}).$$

Therefore, $((f_1',g_1'),\varphi_1')$ and $((f_1,g_1),\varphi_1)$ are in the same cohomology class in $\mathcal{H}^2_{\mathrm{MDLY}}(\mathfrak{g})$.

**Definition 9** A one-parameter formal deformation $(f_t,g_t,\varphi_t)$ of modified $\lambda$-differential Lie-Yamaguti algebra $(\mathfrak{g},[\cdot,\cdot],\{\cdot,\cdot,\cdot\},\varphi)$ is said to be trivial if the deformation $(f_t,g_t,\varphi_t)$ is equivalent to the undeformed one $(f_0,g_0,\varphi)$.

**Definition 10** A modified $\lambda$-differential Lie-Yamaguti algebra $(\mathfrak{g},[\cdot,\cdot],\{\cdot,\cdot,\cdot\},\varphi)$ is said to be rigid if every one-parameter formal deformation of $\mathfrak{g}$ is a trivial deformation.

**Theorem 2** Let $(\mathfrak{g},[\cdot,\cdot],\{\cdot,\cdot,\cdot\},\varphi)$ be a modified $\lambda$-differential Lie-Yamaguti algebra. If $\mathcal{H}^2_{\mathrm{MDLY}}(\mathfrak{g})=0$, then $(\mathfrak{g},[\cdot,\cdot],\{\cdot,\cdot,\cdot\},\varphi)$ is rigid.

**Proof** Let $(f_t,g_t,\varphi_t)$ be a one-parameter formal deformation of $(\mathfrak{g},[\cdot,\cdot],\{\cdot,\cdot,\cdot\},\varphi)$. From Proposition 4, $((f_1,g_1),\varphi_1)$ is a 2-cocycle. By $\mathcal{H}^2_{\mathrm{MDLY}}(\mathfrak{g})=0$, then there exists a 1-cochain

$$\Psi_1\in\mathcal{C}^1_{\mathrm{MDLY}}(\mathfrak{g})=\mathcal{C}^1_{\mathrm{LY}}(\mathfrak{g}),$$

such that $((f_1,g_1),\varphi_1)=\partial^1(\Psi_1)$, that is, $f_1=\delta^1_{\mathrm{I}}(\Psi_1)$, $g_1=\delta^1_{\mathrm{II}}(\Psi_1)$ and $\varphi_1=-\Phi^1(\Psi_1)$.

Setting $\Psi_t = \mathrm{Id}_{\mathfrak{g}} - \Psi_1 t$, we get a deformation $\left(\bar{f}_t, \bar{g}_t, \bar{\varphi}_t\right)$, where

$$\bar{f}_t = \Psi_t^{-1} \circ f_t \circ \left(\Psi_t \otimes \Psi_t\right), \qquad \bar{g}_t = \Psi_t^{-1} \circ g_t \circ \left(\Psi_t \otimes \Psi_t \otimes \Psi_t\right), \qquad \bar{\varphi}_t = \Psi_t^{-1} \circ \varphi_t \circ \Psi_t.$$

It can be easily verify that $\bar{f}_1 = 0, \bar{g}_1 = 0, \bar{\varphi}_1 = 0$. Then

$$\bar{f}_t = f_0 + \bar{f}_2 t^2 + \cdots, \qquad \bar{g}_t = g_0 + \bar{g}_2 t^2 + \cdots, \qquad \varphi_t = \varphi_0 + \bar{\varphi}_2 t^2 + \cdots.$$

By eqs. (7)-(13), we see that $\left(\left(\bar{f}_2, \bar{g}_2\right), \bar{\varphi}_2\right)$ is still a 2-cocyle, so by induction, we can show that $\left(f_t, g_t, \varphi_t\right)$ is equivalent to the trivial one-parameter deformation $\left(f_0, g_0, \varphi\right)$. Therefore, $\left(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \varphi\right)$ is rigid.

## 4 Abelian extensions of modified $\lambda$-differential Lie-Yamaguti algebras

In this section, we consider Abelian extensions of modified $\lambda$-differential Lie-Yamaguti algebras. We prove that any Abelian extension of a modified $\lambda$-differential Lie-Yamaguti algebra has a representation and a 2-cocycle. It is further proved that they are classified by the second cohomology.

Let $\left(\mathfrak{g}, \varphi\right) = \left(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \varphi\right)$ be a modified $\lambda$-differential Lie-Yamaguti algebra and $\left(V, \varphi_V\right)$ be an Abelian modified $\lambda$-differential Lie-Yamaguti algebra, i. e. , the Lie-Yamaguti algebra brackets $[\cdot,\cdot]_V = 0$ and $\{\cdot,\cdot,\cdot\}_V = 0$.

**Definition 11** An Abelian extension $\left(\hat{\mathfrak{g}}, [\cdot,\cdot]_{\hat{\mathfrak{g}}}, \{\cdot,\cdot,\cdot\}_{\hat{\mathfrak{g}}}, \hat{\varphi}\right)$ of $\left(\mathfrak{g}, \varphi\right)$ by $\left(V, \varphi_V\right)$ is a short exact sequence of morphisms of modified $\lambda$-differential Lie-Yamaguti algebras

$$0 \longrightarrow \left(V, \varphi_V\right) \xrightarrow{i} \left(\hat{\mathfrak{g}}, [\cdot,\cdot]_{\hat{\mathfrak{g}}}, \{\cdot,\cdot,\cdot\}_{\hat{\mathfrak{g}}}, \hat{\varphi}\right) \xrightarrow{p} \left(\mathfrak{g}, \varphi\right) \longrightarrow 0$$

where $V$ is an Abelian ideal of $\hat{\mathfrak{g}}$, i. e. , $\varphi_V(u) = \hat{\varphi}(u), \left[u, v\right]_{\hat{\mathfrak{g}}} = 0 = \{\cdot, u, v\}_{\hat{\mathfrak{g}}}, \forall u, v \in V$.

**Definition 12** A section of an Abelian extension $\left(\hat{\mathfrak{g}}, [\cdot,\cdot]_{\hat{\mathfrak{g}}}, \{\cdot,\cdot,\cdot\}_{\hat{\mathfrak{g}}}, \hat{\varphi}\right)$ of $\left(\mathfrak{g}, \varphi\right)$ by $\left(V, \varphi_V\right)$ is a linear map $s{:}\mathfrak{g} \longrightarrow \hat{\mathfrak{g}}$ such that $p \circ s = \mathrm{Id}_{\mathfrak{g}}$.

**Definition 13** Let $\left(\hat{\mathfrak{g}}_1, [\cdot,\cdot]_{\hat{\mathfrak{g}}_1}, \{\cdot,\cdot,\cdot\}_{\hat{\mathfrak{g}}_1}, \hat{\varphi}_1\right)$ and $\left(\hat{\mathfrak{g}}_2, [\cdot,\cdot]_{\hat{\mathfrak{g}}_2}, \{\cdot,\cdot,\cdot\}_{\hat{\mathfrak{g}}_2}, \hat{\varphi}_2\right)$ be two Abelian extensions of $\left(\mathfrak{g}, \varphi\right)$ by $\left(V, \varphi_V\right)$. They are said to be equivalent if there is an isomorphism of modified $\lambda$-differential Lie-Yamaguti algebras $\eta{:}\left(\hat{\mathfrak{g}}_1, [\cdot,\cdot]_{\hat{\mathfrak{g}}_1}, \{\cdot,\cdot,\cdot\}_{\hat{\mathfrak{g}}_1}, \hat{\varphi}_1\right) \rightarrow \left(\hat{\mathfrak{g}}_2, [\cdot,\cdot]_{\hat{\mathfrak{g}}_2}, \{\cdot,\cdot,\cdot\}_{\hat{\mathfrak{g}}_2}, \hat{\varphi}_2\right)$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 \longrightarrow & \left(V, \varphi_V\right) & \xrightarrow{i_1} & \left(\hat{\mathfrak{g}}_1, [\cdot,\cdot]_{\hat{\mathfrak{g}}_1}, \{\cdot,\cdot,\cdot\}_{\hat{\mathfrak{g}}_1}, \hat{\varphi}_1\right) & \xrightarrow{p_1} & \left(\mathfrak{g}, \varphi\right) & \longrightarrow 0 \\ & \Big\| & & \Big\downarrow \eta & & \Big\| & \\ 0 \longrightarrow & \left(V, \varphi_V\right) & \xrightarrow{i_2} & \left(\hat{\mathfrak{g}}_2, [\cdot,\cdot]_{\hat{\mathfrak{g}}_2}, \{\cdot,\cdot,\cdot\}_{\hat{\mathfrak{g}}_2}, \hat{\varphi}_2\right) & \xrightarrow{p_2} & \left(\mathfrak{g}, \varphi\right) & \longrightarrow 0 \end{array} \tag{22}$$

The set of all equivalence classes of Abelian extensions of $\left(\mathfrak{g}, \varphi\right)$ by $\left(V, \varphi_V\right)$ is denoted by $\mathrm{Ext}_{ae}\left(\mathfrak{g}, V\right)$.

Now for an Abelian extension $\left(\hat{\mathfrak{g}}, [\cdot,\cdot]_{\hat{\mathfrak{g}}}, \{\cdot,\cdot,\cdot\}_{\hat{\mathfrak{g}}}, \hat{\varphi}\right)$ of $\left(\mathfrak{g}, \varphi\right)$ by $\left(V, \varphi_V\right)$ with a section $s{:}\, \mathfrak{g} \rightarrow \hat{\mathfrak{g}}$, we define linear maps $\rho{:}\, \mathfrak{g} \longrightarrow \mathrm{End}(V)$ and $\theta, D{:}\, \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathrm{End}(V)$ by

$$\rho(x)u \coloneqq \left[s(x), u\right]_{\hat{\mathfrak{g}}}, \qquad \theta\left(x, y\right)u \coloneqq \{u, s(x), s\left(y\right)\}_{\hat{\mathfrak{g}}}, \quad \forall x, y \in \mathfrak{g}, u \in V.$$

In particular, $D\left(x, y\right)u = \{s(x), s\left(y\right), u\}_{\hat{\mathfrak{g}}}$.

**Proposition 6** With the above notations, $\left(V; \rho, \theta, D, \varphi_V\right)$ is a representation of the modified $\lambda$-differential Lie-Yamaguti algebra $\left(\mathfrak{g}, \varphi\right)$.

**Proof** In view of Zhang et al. (2015), $\left(V; \rho, \theta, D\right)$ is a representation of the Lie-Yamaguti algebra $\mathfrak{g}$. Further, for any $x, y \in \mathfrak{g}$ and $u \in V$, $\hat{\varphi}s(x) - s\left(\varphi(x)\right) \in V$ means that $\rho\left(\hat{\varphi}s(x)\right)u = \rho\left(s\varphi(x)\right)u$, $\theta\left(\hat{\varphi}s(x), \hat{\varphi}s\left(y\right)\right)u =$

$\theta\big(s\varphi(x), s\varphi(y)\big)u$. Therefore, we have

$$\begin{aligned}\varphi_V\big(\rho(x)u\big) &= \varphi_V\big[s(x), u\big]_{\hat{\mathfrak{g}}} = \hat{\varphi}\big[s(x), u\big]_{\hat{\mathfrak{g}}} = \big[\hat{\varphi}s(x), u\big]_{\hat{\mathfrak{g}}} + \big[s(x), \hat{\varphi}(u)\big]_{\hat{\mathfrak{g}}} + \lambda\big[s(x), u\big]_{\hat{\mathfrak{g}}}\\ &= \rho\big(\varphi(x)\big)u + \rho(x)\varphi_V(u) + \lambda\rho(x)u,\\ \varphi_V\big(\theta(x,y)u\big) &= \varphi_V\{u, s(x), s(y)\}_{\hat{\mathfrak{g}}} = \hat{\varphi}\{u, s(x), s(y)\}_{\hat{\mathfrak{g}}}\\ &= \{\hat{\varphi}(u), s(x), s(y)\}_{\hat{\mathfrak{g}}} + \{u, \hat{\varphi}s(x), s(y)\}_{\hat{\mathfrak{g}}} + \{u, s(x), \hat{\varphi}s(y)\}_{\hat{\mathfrak{g}}} + 2\lambda\{u, s(x), s(y)\}_{\hat{\mathfrak{g}}}\\ &= \theta(x,y)\varphi_V(u) + \theta\big(\varphi(x), y\big)u + \theta\big(x, \varphi(y)\big)u + 2\lambda\theta(x,y)u.\end{aligned}$$

Hence, $\big(V;\rho,\theta,D,\varphi_V\big)$ is a representation of $\big(\mathfrak{g},\varphi\big)$.

We further define linear maps $\nu: \mathfrak{g}\times\mathfrak{g}\to V$, $\psi: \mathfrak{g}\times\mathfrak{g}\times\mathfrak{g}\to V$ and $\chi: \mathfrak{g}\to V$ respectively by

$$\begin{aligned}\nu(x,y) &= \big[s(x), s(y)\big]_{\hat{\mathfrak{g}}} - s\big([x,y]\big),\\ \psi(x,y,z) &= \{s(x), s(y), s(z)\}_{\hat{\mathfrak{g}}} - s\big(\{x,y,z\}\big),\\ \chi(a) &= \hat{\varphi}\big(s(x)\big) - s\big(\varphi(x)\big), \quad \forall x,y,z\in\mathfrak{g}.\end{aligned}$$

We transfer the modified $\lambda$-differential Lie-Yamaguti algebra structure on $\hat{\mathfrak{g}}$ to $\mathfrak{g}\oplus V$ by endowing $\mathfrak{g}\oplus V$ with multiplications $[\cdot,\cdot]_\nu$, $\{\cdot,\cdot,\cdot\}_\psi$ and a modified $\lambda$-differential operator $\varphi_\chi$ defined by

$$\begin{aligned}[x+u, y+v]_\nu &= [x,y] + \rho(x)v - \rho(y)u + \nu(x,y),\\ \{x+u, y+v, z+w\}_\psi &= \{x,y,z\} + \theta(y,z)u - \theta(x,z)v + D(x,y)w + \psi(x,y,z),\\ \varphi_\chi(x+u) &= \varphi(x) + \chi(x) + \varphi_V(u), \quad \forall x,y,z\in\mathfrak{g},\ u,v,w\in V.\end{aligned}$$

**Proposition 7** The 4-tuple $\big(\mathfrak{g}\oplus V, [\cdot,\cdot]_\nu, \{\cdot,\cdot,\cdot\}_\psi, \varphi_\chi\big)$ is a modified $\lambda$-differential Lie-Yamaguti algebra if and only if $\big((\nu,\psi),\chi\big)$ is a 2-cocycle of the modified $\lambda$-differential Lie-Yamaguti algebra $\big(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \varphi\big)$ with the coefficient in $\big(V;\rho,\theta,D,\varphi_V\big)$. In this case,

$$0\longrightarrow (V,\varphi_V)\xrightarrow{i}\big(\mathfrak{g}\oplus V, [\cdot,\cdot]_\nu, \{\cdot,\cdot,\cdot\}_\psi, \varphi_\chi\big)\xrightarrow{p}(\mathfrak{g},\varphi)\longrightarrow 0$$

is an Abelian extension.

**Proof** In view of Zhang et al. (2015), $\big(\mathfrak{g}\oplus V, [\cdot,\cdot]_\nu, \{\cdot,\cdot,\cdot\}_\psi\big)$ is a Lie-Yamaguti algebra if and only if $\delta^2(\nu,\psi) = 0$. The map $\varphi_\chi$ is a modified $\lambda$-differential operator on $\big(\mathfrak{g}\oplus V, [\cdot,\cdot]_\nu, \{\cdot,\cdot,\cdot\}_\psi\big)$ if and only if

$$\begin{aligned}\varphi_\chi[x+u, y+v]_\nu &= \big[\varphi_\chi(x+u), y+v\big]_\nu + \big[x+u, \varphi_\chi(y+v)\big]_\nu + \lambda[x+u, y+v]_\nu,\\ \varphi_\chi\{x+u, y+v, z+w\}_\psi &= \{\varphi_\chi(x+u), y+v, z+w\}_\psi + \{x+u, \varphi_\chi(y+v), z+w\}_\psi\\ &\quad + \{x+u, y+v, \varphi_\chi(z+w)\}_\psi) + 2\lambda\{x+u, y+v, z+w\}_\psi,\end{aligned}$$

for any $x,y,z\in\mathfrak{g}$ and $u,v,w\in V$. Further, we get that the above equations are equivalent to the following equations:

$$\chi[x,y] + \varphi_V\big(\nu(x,y)\big) = \nu\big(\varphi(x), y\big) + \nu\big(x, \varphi(y)\big) + \lambda\nu(x,y) + \rho(x)\chi(y) - \rho(y)\chi(x), \tag{23}$$

$$\begin{aligned}\chi\{x,y,z\} + \varphi_V\big(\psi(x,y,z)\big) &= \psi\big(\varphi(x), y, z\big) + \psi\big(x, \varphi(y), z\big) + \psi\big(x, y, \varphi(z)\big) + 2\lambda\psi(x,y,z)\\ &\quad + \theta(y,z)\chi(x) - \theta(x,z)\chi(y) + D(x,y)\chi(z).\end{aligned} \tag{24}$$

Using eqs. (23) and (24), we get $\delta^1_{\mathrm{I}}(\chi) + \Phi^2_{\mathrm{I}}(\nu) = 0$ and $\delta^1_{\mathrm{II}}(\chi) + \Phi^2_{\mathrm{II}}(\psi) = 0$ respectively. Therefore, $\partial^2\big((\nu,\psi),\chi\big) = \big(\delta^2(\nu,\psi), \delta^1(\chi) + \Phi^2(\nu,\psi)\big) = 0$, that is, $\big((\nu,\psi),\chi\big)$ is a 2-cocycle.

Conversely, if $\big((\nu,\psi),\chi\big)$ is a 2-cocycle of the modified $\lambda$-differential Lie-Yamaguti algebra $\big(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \varphi\big)$ with the coefficient in $\big(V;\rho,\theta,D,\varphi_V\big)$, then we have $\partial^2\big((\nu,\psi),\chi\big) = \big(\delta^2(\nu,\psi),\ \delta^1(\chi) + \Phi^2(\nu,\psi)\big) = 0$, in which $\delta^2(\nu,\psi) = 0$, eqs. (23) and (24) hold. So $\big(\mathfrak{g}\oplus V, [\cdot,\cdot]_\nu, \{\cdot,\cdot,\cdot\}_\psi, \varphi_\chi\big)$ is a modified $\lambda$-differential Lie-Yamaguti algebra.

**Proposition 8** Let $\left(\hat{\mathfrak{g}},[\cdot,\cdot]_{\hat{\mathfrak{g}}},\{\cdot,\cdot,\cdot\}_{\hat{\mathfrak{g}}},\hat{\varphi}\right)$ be an Abelian extension of $(\mathfrak{g},\varphi)$ by $(V,\varphi_V)$ and $s\colon\mathfrak{g}\to\hat{\mathfrak{g}}$ a section. If $\left((\nu,\psi),\chi\right)$ is a 2-cocycle constructed using the section $s$, then its cohomology class does not depend on the choice of $s$.

**Proof** Let $s_1,s_2\colon\mathfrak{g}\to\hat{\mathfrak{g}}$ be two distinct sections, then we get two corresponding 2-cocycles $\left((\nu_1,\psi_1),\chi_1\right)$ and $\left((\nu_2,\psi_2),\chi_2\right)$ respectively. Define a linear map $\omega\colon\mathfrak{g}\to V$ by $\omega(x)=s_1(x)-s_2(x)$ for $x\in\mathfrak{g}$. Further,

$$\begin{aligned}\nu_1(x,y)&=\left[s_1(x),s_1(y)\right]_{\hat{\mathfrak{g}}}-s_1[x,y]=\left[s_2(x)+\omega(x),s_2(y)+\omega(y)\right]_{\hat{\mathfrak{g}}}-s_2([x,y])-\omega[x,y]\\&=\left[s_2(x),s_2(y)\right]_{\hat{\mathfrak{g}}}+\rho(x)\omega(y)-\rho(y)\omega(x)-s_2([x,y])-\omega[x,y]=\nu_2(x,y)+\delta_{\mathrm{I}}^1\omega(x,y),\end{aligned}$$

$$\begin{aligned}\psi_1(x,y,z)&=\{s_1(x),s_1(y),s_1(z)\}_{\hat{\mathfrak{g}}}-s_1\{x,y,z\}\\&=\{s_2(x)+\omega(x),s_2(y)+\omega(y),s_2(z)+\omega(z)\}_{\hat{\mathfrak{g}}}-s_2\{x,y,z\}-\omega\{x,y,z\}\\&=\{s_2(x),s_2(y),s_2(z)\}_{\hat{\mathfrak{g}}}+\theta(y,z)\omega(x)-\theta(x,z)\omega(y)+D(x,y)\omega(z)-\omega\{x,y,z\}-s_2\{x,y,z\}\\&=\psi_2(x,y,z)+\delta_{\mathrm{II}}^1\omega(x,y,z),\end{aligned}$$

$$\begin{aligned}\chi_1(x)&=\hat{\varphi}s_1(x)-s_1\varphi(x)=\hat{\varphi}\left(s_2(x)+\omega(x)\right)-\left(s_2\varphi(x)+\omega\varphi(x)\right)\\&=\hat{\varphi}s_2(x)-s_2\varphi(x)+\hat{\varphi}\omega(x)-\omega\left(\varphi(x)\right)=\chi_2(x)+\varphi_V\omega(x)-\omega\left(\varphi(x)\right)=\chi_2(x)-\Phi^1\omega(x).\end{aligned}$$

So $\left((\nu_1,\psi_1),\chi_1\right)-\left((\nu_2,\psi_2),\chi_2\right)=\left(\delta^1\omega,-\Phi^1\omega\right)=\partial^1(\omega)\in\mathcal{B}^2_{\mathrm{MDLY}}(\mathfrak{g},V)$, that is $\left((\nu_1,\psi_1),\chi_1\right)$ and $\left((\nu_2,\psi_2),\chi_2\right)$ are in the same cohomology class in $\mathcal{H}^2_{\mathrm{MDLY}}(\mathfrak{g},V)$.

**Theorem 3** Abelian extensions of a modified $\lambda$-differential Lie-Yamaguti algebra $(\mathfrak{g},\varphi)$ by $(V,\varphi_V)$ are classified by the second cohomology group $\mathcal{H}^2_{\mathrm{MDLY}}(\mathfrak{g},V)$ of $(\mathfrak{g},\varphi)$ with coefficients in the representation $(V;\rho,\theta,D,\varphi_V)$. Namely, there exists a bijection $\mathrm{Ext}_{ae}(\mathfrak{g},V)\simeq\mathcal{H}^2_{\mathrm{MDLY}}(\mathfrak{g},V)$.

**Proof** Suppose $\left(\hat{\mathfrak{g}}_1,[\cdot,\cdot]_{\hat{\mathfrak{g}}_1},\{\cdot,\cdot,\cdot\}_{\hat{\mathfrak{g}}_1},\hat{\varphi}_1\right)$ and $\left(\hat{\mathfrak{g}}_2,[\cdot,\cdot]_{\hat{\mathfrak{g}}_2},\{\cdot,\cdot,\cdot\}_{\hat{\mathfrak{g}}_2},\hat{\varphi}_2\right)$ are equivalent Abelian extensions of $(\mathfrak{g},\varphi)$ by $(V,\varphi_V)$ with the associated isomorphism $\eta\colon\left(\hat{\mathfrak{g}}_1,[\cdot,\cdot]_{\hat{\mathfrak{g}}_1},\{\cdot,\cdot,\cdot\}_{\hat{\mathfrak{g}}_1},\hat{\varphi}_1\right)\to\left(\hat{\mathfrak{g}}_2,[\cdot,\cdot]_{\hat{\mathfrak{g}}_2},\{\cdot,\cdot,\cdot\}_{\hat{\mathfrak{g}}_2},\hat{\varphi}_2\right)$ such that the diagram in (22) is commutative. Let $s_1$ be a section of $\left(\hat{\mathfrak{g}}_1,[\cdot,\cdot]_{\hat{\mathfrak{g}}_1},\{\cdot,\cdot,\cdot\}_{\hat{\mathfrak{g}}_1},\hat{\varphi}_1\right)$. As $p_2\circ\eta=p_1$, we have

$$p_2\circ(\eta\circ s_1)=(p_2\circ\eta)\circ s_1=p_1\circ s_1=\mathrm{Id}_{\mathfrak{g}}.$$

So the map $\eta\circ s_1$ is a section of $\left(\hat{\mathfrak{g}}_2,[\cdot,\cdot]_{\hat{\mathfrak{g}}_2},\{\cdot,\cdot,\cdot\}_{\hat{\mathfrak{g}}_2},\hat{\varphi}_2\right)$. Denote $s_2\coloneqq\eta\circ s_1$. Since $\eta$ is an isomorphism of modified $\lambda$-differential Lie-Yamaguti algebras such that $\eta|_V=\mathrm{Id}_V$, we have

$$\begin{aligned}\nu_2(x,y)&=\left[s_2(x),s_2(y)\right]_{\hat{\mathfrak{g}}_2}-s_2([x,y])=\left[\eta(s_1(x)),\eta(s_1(y))\right]_{\hat{\mathfrak{g}}_2}-\eta\left(s_1([x,y])\right)\\&=\eta\left(\left[s_1(x),s_1(y)\right]_{\hat{\mathfrak{g}}_1}-s_1([x,y])\right)=\eta\left(\nu_1(x,y)\right)=\nu_1(x,y),\end{aligned}$$

$$\psi_2(x,y,z)=\{s_2(x),s_2(y),s_2(z)\}_{\hat{\mathfrak{g}}_2}-s_2(\{x,y,z\})=\eta\left(\{s_1(x),s_1(y),s_1(z)\}_{\hat{\mathfrak{g}}_1}-s_1\{x,y,z\}\right)=\psi_1(x,y,z),$$

$$\chi_2(x)=\hat{\varphi}\left(s_2(x)\right)-s_2\left(\varphi(x)\right)=\hat{\varphi}\left(\eta\circ s_1(x)\right)-\eta\circ s_1\left(\varphi(x)\right)=\hat{\varphi}\left(s_1(x)\right)-s_1\left(\varphi(x)\right)=\chi_1(x).$$

So all equivalent Abelian extensions give rise to the same element in $\mathcal{H}^2_{\mathrm{MDLY}}(\mathfrak{g},V)$. Therefore, there is a well-defined map $\Gamma\colon\mathrm{Ext}_{ae}(\mathfrak{g},V)\to\mathcal{H}^2_{\mathrm{MDLY}}(\mathfrak{g},V)$.

Conversely, given two cohomologous 2-cocycles $\left((\nu_1,\psi_1),\chi_1\right)$ and $\left((\nu_2,\psi_2),\chi_2\right)$ in $\mathcal{H}^2_{\mathrm{MDLY}}(\mathfrak{g},V)$, we can construct two Abelian extensions $\left(\mathfrak{g}\oplus V,[\cdot,\cdot]_{\nu_1},\{\cdot,\cdot,\cdot\}_{\psi_1},\varphi_{\chi_1}\right)$ and $\left(\mathfrak{g}\oplus V,[\cdot,\cdot]_{\nu_2},\{\cdot,\cdot,\cdot\}_{\psi_2},\varphi_{\chi_2}\right)$ via Proposition 7. Then there is a linear map $\omega\colon\mathfrak{g}\to V$ such that

$$\left((\nu_1,\psi_1),\chi_1\right)-\left((\nu_2,\psi_2),\chi_2\right)=\partial^1(\omega)=\left(\delta^1\omega,-\Phi^1\omega\right).$$

Define a linear map $\eta_\omega\colon\mathfrak{g}\oplus V\to\mathfrak{g}\oplus V$ by $\eta_\omega(x+u)\coloneqq x+\omega(x)+u$ for $x\in\mathfrak{g},u\in V$. Then we have that $\eta_\omega$

is an isomorphism of these two Abelian extensions. Therefore, $\Gamma: \mathrm{Ext}_{ae}(\mathfrak{g}, V) \to \mathcal{H}^2_{\mathrm{MDLY}}(\mathfrak{g}, V)$ is bijective. This completes the proof.

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# 修正$\lambda$-微分Lie-Yamaguti代数的形变和扩张


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**摘　要：**考虑了修正$\lambda$-微分Lie-Yamaguti代数，其由一个Lie-Yamaguti代数和一个修正$\lambda$-微分算子组成. 首先我们引入修正$\lambda$-微分Lie-Yamaguti代数的表示. 此外，我们建立了系数在表示中的修正$\lambda$-微分Lie-Yamaguti代数的上同调. 最后，我们利用第二上同调群研究了修正$\lambda$-微分Lie-Yamaguti代数的单参数形式变形和Abelian扩张.



**关键词：**Lie-Yamaguti代数；修正$\lambda$-微分算子；表示和上同调；单参数形式变形；Abelian扩张



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